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Rimming flows within a rotating horizontal cylinder: asymptotic analysis of the thin-film lubrication equations and stability of their solutions

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Abstract. It is well-known that a standard lubrication analysis of the equations of motion in thin liquid films coating the inside surface of a rotating horizontal cylinder leads, under creeping-flow conditions, to a cubic equation for the film thickness profile which, depending on the fluid properties of the liquid, the speed of rotation and the fill fraction F, has either (a) a continuous, symmetric (homogeneous) solution; (b) a solution containing a shock; or (c) no solution below a certain speed. By means of an asymptotic analysis of the recently proposed "modified lubrication equation" (MLE) [M. Tirumkudulu and A. Acrivos, Phys. Fluid 13 (2000) 14–19], it is shown that the solutions of the cubic equation referred to above correctly describe the film-thickness profiles although, when shocks are involved, under exceedingly restrictive conditions, typically $F \sim 10^{-3}$ or less. In addition, using the MLE, the linear stability of these film profiles is investigated and it is shown that: the "homogeneous" profiles are neutrally stable if surface-tension effects are neglected but, if the latter are retained, the films are asymptotically *stable* to two-dimensional disturbances and *unstable* to axial disturbances; on the other hand, the non-homogeneous profiles are always asymptotically *stable*, thus confirming results given earlier [T.B. Benjamin, W.G. Pritchard, and S.J. Tavener (preprint, 1993)] on the basis of the standard lubrication analysis.

Key words: asymptotic analysis, horizontal cylinder, lubrication equations, rimming flows, thin films

1. Introduction

Coating flows within a rotating horizontal cylinder, often referred to as *rimming flows*, have received an increasing degree of attention in recent years on account of the fascinating variety of flow patterns that are encountered and the intriguing mathematical properties of the equations which have been developed to model the observed phenomena. To date, most of the attention has been directed to the case of thin films for which the well-known lubrication approximation can be invoked to reduce the governing equations of viscous-flow hydrodynamics to the particularly simple form

$$\nu \frac{\partial^2 \hat{v}}{\partial y^2} = g \cos\theta, \tag{1}$$

provided that inertial and surface-tension effects are negligible. Here, $\hat{v}$ is the velocity component in the angular direction, θ is the angular coordinate, ν is the kinematic viscosity, g is the gravitational constant, $y = R - r$ is the distance from the rotating cylindrical boundary and r is the radial coordinate with R being the inner radius of the cylinder. There are two boundary conditions for this equation: the no-slip condition $\hat{v} = \Omega R$, where Ω is the angular velocity of the cylinder; and $\partial \hat{v}/\partial y = 0$, the condition of vanishing shear stress at the free surface $y = h$ (*cf.* Figure 1). Hence, the solution of Equation (1) is

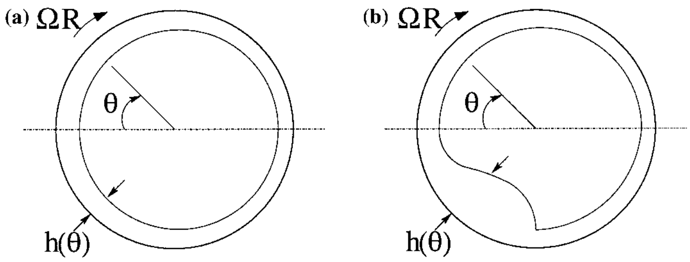

Figure 1. Sketch of the liquid film profiles for (a) a homogeneous film, $\beta < \beta^*$; (b) an inhomogeneous film, $\beta > \beta^*$, where θ is the angular coordinate, Ω is the angular velocity of the cylinder, R is the inner radius of the cylinder and $h(\theta)$ is the film thickness.

$$\hat{v} = \Omega R - \frac{g}{\nu}\cos\theta\left(hy - \frac{1}{2}y^2\right). \tag{2}$$

The approximations leading to (1) and (2) require that $h \ll R$ and that $|\partial h/\partial\theta|$ be at most $O(h)$. On integrating Equation (2) with respect to y from 0 to h, we then obtain the total volumetric flow rate within the film per unit axial distance,

$$Q = \Omega R h - \frac{1}{3}\frac{g}{\nu}h^3\cos\theta. \tag{3}$$

It proves convenient to introduce the important dimensionless parameter

$$\alpha \equiv \sqrt{\frac{\Omega\nu}{gR}} \tag{4}$$

and to define a dimensionless thickness η and a dimensionless flow rate q by

$$\eta \equiv h/\alpha R \qquad \text{and} \qquad q \equiv Q/\alpha\Omega R^2,$$

so that (3) becomes

$$q = \eta - \frac{1}{3}\eta^3\cos\theta, \tag{5}$$

to be referred to as the standard lubrication equation (SLE), first studied in some detail by Moffatt [1]. In addition to being a real and positive root of (5), η is required to be periodic in 2π and to satisfy the overall liquid-volume-conservation condition, in the thin-film approximation,

$$F = \frac{1}{\pi R}\int_{-\pi}^{\pi} h\,\mathrm{d}\theta = \frac{\alpha}{\pi}\int_{-\pi}^{\pi}\eta\,\mathrm{d}\theta, \tag{6}$$

where F is the fill fraction (*i.e.*, the fractional cross-sectional area occupied by the liquid). Besides requiring that the liquid film be everywhere thin relative to R, the radius of the cylinder, Equation (5) presupposes steady-state (hence q is independent of θ), plus, as was said earlier, negligible inertia and surface-tension effects, *i.e.*, vanishingly small Reynolds number and inverse Bond number $\gamma \equiv \sigma/\rho g R^2$, where σ is the surface tension and ρ is the density of the fluid. Under these conditions, the remaining two independent dimensionless groups, F and α, can be combined into a single parameter β,

$$\beta \equiv F/\alpha = \frac{1}{\pi}\int_{-\pi}^{\pi}\eta\,\mathrm{d}\theta, \tag{7}$$

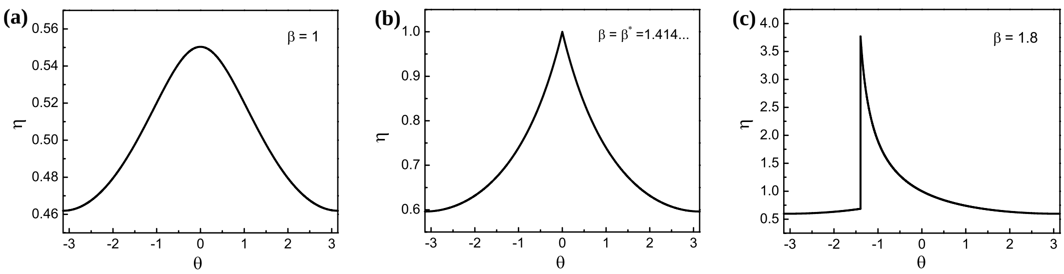

Figure 2. The solutions of the standard lubrication equation (SLE) (*i.e.*, Equation (5) subject to (7)) for (a) $\beta = 1$; (b) $\beta = \beta^* = 1{\cdot}414\ldots$; (c) $\beta = 1{\cdot}8$.

the value of which implicitly determines q and therefore the solution of (5) subject to (7) plus the periodicity condition $\eta(\theta + 2\pi) = \eta(\theta)$.

It is well-known [1–6] that the solution to the cubic Equation (5) depends critically on the value of β. Briefly, when $0 < \beta < \beta^* = 1{\cdot}4142\ldots$ (corresponding to $0 < q < 2/3$), η is symmetric about $\theta = 0$ and everywhere continuous. This is referred to in the literature as a *homogeneous* film and a typical such profile for $\beta = 1{\cdot}0$ ($q = 0{\cdot}485$) is shown in Figure 2(a). When $\beta = \beta^*$, in which case q attains its maximum value 2/3, the film is still continuous but has a discontinuous slope at $\theta = 0$ equal to $\pm 1/\sqrt{6}$. This profile is shown in Figure 2(b). When $\beta^* < \beta < \beta^{**} = 2{\cdot}21\ldots$ and $q = 2/3$, a physically acceptable solution to (5) can still be constructed with, however, a discontinuity at $\theta = \theta^* < 0$, the value of which is determined by β. Such a profile for $\beta = 1{\cdot}8$ is shown in Figure 2(c). Moreover, $\theta^* = -\pi/2$ when $\beta = \beta^{**}$, and when β exceeds β^{**} no solution exists to Equation (5) subject to (7) which is everywhere real and positive. The discontinuity in the film-thickness profile is disquieting of course, since its existence is incompatible with the assumptions underlying the lubrication analysis leading to Equation (5). Equally troublesome, is the absence of solutions of (5) subject to (7) when $\beta > \beta^{**}$. This then raises the question as to whether and under what circumstances, the discontinuous solutions of Equation (5) when $\beta > \beta^*$ can faithfully represent the dynamics of the film flow under consideration.

In order to answer this query, a *model* equation was recently proposed [6] which was obtained by simply adding to (5) a term that accounts for the angular variation of the hydrostatic pressure. Here, we briefly discuss the basis for this model:

First, from the lubrication analysis, it can be shown that the leading-order term of the pressure in the thin film is hydrostatic [3,6] hence,

$$p = \alpha(z - \eta)\sin\theta, \tag{8}$$

where $p \equiv \hat{p}/\rho g R$ is the dimensionless pressure with $\hat{p}$ being the dimensional pressure and $z \equiv y/\alpha R$ is the dimensionless distance from the rotating cylindrical boundary. Then, inserting Equation (8) into the θ-momentum equation leads to a new term containing the θ-component of ∇p on the right-hand of (1) in the dimensionless form, *viz.*

$$\frac{\partial^2 v}{\partial z^2} = \cos\theta + \frac{\alpha}{1 - \alpha z}\left\{(z - \eta)\cos\theta - \eta'\sin\theta\right\}, \text{which can be rearranged into} \tag{9}$$

$$\frac{\partial^2 v}{\partial z^2} = \left\{(1 - \alpha\eta)\cos\theta - \alpha\eta'\sin\theta\right\},$$

provided that $\alpha z \ll 1$, where $v \equiv \hat{v}/\Omega R$ is the dimensionless angular component of the velocity. Using the same boundary conditions as before ($v = 1$ at $z = 0$ and $\partial v/\partial z = 0$ at $z = \eta$), the solution of (9) is

$$v = 1 - \left(\eta z - \frac{z^2}{2}\right)\{(1-\alpha\eta)\cos\theta - \alpha\eta'\sin\theta\}. \tag{10}$$

Therefore, in lieu of Equation (5), this modified lubrication analysis (MLA) of [6] leads then to

$$q = \eta - \frac{1}{3}\eta^3\cos\theta + \frac{\alpha}{3}\left\{\eta^4\cos\theta + \eta^3\eta'\sin\theta\right\}, \tag{11}$$

the solution of which must be periodic in 2π and must satisfy the exact overall liquid-volume-conservation condition

$$F = \frac{\alpha}{\pi}\int_{-\pi}^{\pi}\left(1 - \frac{\alpha\eta}{2}\right)\eta \ \mathrm{d}\theta, \tag{12}$$

from which it follows that

$$\beta \equiv \frac{F}{\alpha} = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(1 - \frac{\alpha\eta}{2}\right)\eta \ \mathrm{d}\theta. \tag{13}$$

Equation (11), to be referred to as the modified lubrication equation (MLE), has the useful property that its solution, when the flow is turned off, *i.e.,* when $\alpha = 0$ with F fixed, gives the exact shape of the stagnant pool at the bottom of the cylinder [6]. Thus, Equation (11) subject to (13) has a continuous periodic solution for all α with F fixed (below a critical value of $F = 0{\cdot}36$ as shown in [7]), in contrast to (5) subject to (7) which does not have a solution if $\alpha < F/\beta^{**}$.

In point of fact, Equation (11) is very similar to an expression for q given earlier by Johnson [2] as well as by Benjamin, Pritchard and Tavener [3] (see also [8]) who expanded the variables in the Stokes equations plus the boundary conditions in powers of α and, on retaining all the $O(\alpha)$ terms but discarding those of higher order, arrived at

$$q = \eta - \frac{1}{3}\eta^3\cos\theta + \alpha\left\{\frac{1}{2}\eta^4\cos\theta - \frac{1}{2}\eta^2 + \frac{1}{3}\eta^3\eta'\sin\theta\right\}, \tag{14}$$

subject to (13). As shown in Ref [7], Equation(14) has continuous, periodic solutions for all α when $F \leq 0{\cdot}29$. But, although, as was also shown in [7], the solutions of (14) are, numerically, very close to those of (11) for the same values of α and $F \leq 0{\cdot}29$, our discussion which follows will make use primarily of Equation (11) because of its slighter simplicity relative to (14). We recall that, although (11) and (14) are nothing more than *model* equations rather than rigorous higher-order corrections to (5), the film profiles obtained by solving them numerically were found to be in very good agreement with those determined experimentally [6] or via the numerical solution of the full Stokes equations using the FIDAP software [7].

It is obvious that the solution of either (11) or of (14) subject to (13) now depends on the values of the two independent parameters F and α, rather than merely on their ratio β as was the case with Equation (5) subject to (7). We also note for future use that, if the capillary contribution is included in the thin-film approximation, then, according to [3,8], the expression for the hydrostatic pressure given previously, *i.e.,* Equation (8), is modified into

$$p = \alpha(z - \eta)\sin\theta - \alpha\gamma(\eta + \eta''), \tag{15}$$

where, as noted earlier, $\gamma \equiv \sigma/\rho g R^2$ is the inverse Bond number. We then arrive at either

$$q = \eta - \frac{1}{3}\eta^3\cos\theta + \frac{\alpha}{3}\{\eta^4\cos\theta + \eta^3\eta'\sin\theta + \gamma\eta^3(\eta' + \eta''')\} \tag{16}$$

or

$$q = \eta - \frac{1}{3}\eta^3 \cos\theta + \alpha \left\{ \frac{1}{2}\eta^4 \cos\theta - \frac{1}{2}\eta^2 + \frac{1}{3}\eta^3 \eta' \sin\theta + \frac{1}{3}\gamma \eta^3 (\eta' + \eta''') \right\}, \tag{17}$$

which reduce, respectively, to (11) or (14) if $\gamma \ll 1$ and both η' and η''' are $O(\eta)$. Some of the consequences of retaining the full nonlinear capillary terms in (15) have been examined in a recent study [9], but, here, we shall suppose that either the surface-tension effects are entirely negligible ($\gamma = 0$) or that only the linear approximation shown in (15) needs to be retained to a first approximation.

Offhand, it might seem that (5) and (7), the equations of the standard lubrication analysis, can be recovered from (11), or (14), subject to (13) by simply setting $\alpha = 0$ with F fixed. Clearly, however, this cannot possibly work out, given that, as was pointed out earlier, $\alpha = 0$ (implying $\Omega = 0$) leads to a stagnant pool having a shape that is incompatible with (5), which anyhow, has no solution given that, in this case, $\beta = \infty$. The other possibility, $F \to 0$ with α fixed, is much too restrictive, given that since, in this case, $\beta \to 0$, the film has a uniform thickness ($h/R = F/2$). Nevertheless, we shall show, starting from (11), that, as $\alpha \to 0$, the asymmetric solution of (5) does in fact represent asymptotically the liquid-film profiles *everywhere*, except at the point of discontinuity, but under extremely restrictive conditions, specifically, for *fixed* β and for values of F which are, typically, almost vanishingly small. Then, again starting from (11), we shall examine the stability of these two-dimensional flows to both two-dimensional as well as axial disturbances. We confirm the results obtained earlier by Benjamin *et al.* [3] on the basis of the standard lubrication theory, Equation (5), and show that the *inhomogeneous* profiles ($\beta > \beta^*$) of (11) are asymptotically stable to the small disturbances referred to above and that the *homogeneous* profiles ($\beta < \beta^*$) of (11) are only neutrally stable if surface-tension effects are ignored. We shall further show, however, that, if the latter are included in the analysis, the homogeneous profiles are: on the one hand, also asymptotically stable (*but very weakly so for* $\beta \to 0$) to small *two-dimensional* disturbances, but, on the other hand and surprisingly to us, are asymptotically *unstable* to axial disturbances. This last result, which appears to be new, might explain the "rings" that have been reported in the literature (*cf.* Figure 2 in [10]) as having been observed under certain conditions.

2. Film profiles for $\alpha \to 0$ and fixed β

As was said earlier, the solutions of (11) and (13) when $\alpha \to 0$ with F fixed bear no relation to those of the standard lubrication analysis, especially considering that the latter does not have a solution for $\alpha < F/\beta^{**}$. We now turn to the case $\alpha \to 0$ with $\beta \equiv F/\alpha$ fixed and examine under what conditions, if any, the solutions of (11) and (13) asymptote to those of (5) and (7). We shall examine separately the three cases: (a) $0 < \beta < \beta^*$; (b) $\beta^* < \beta < \beta^{**}$; and (c) $\beta > \beta^{**}$.

2.1. Case a ($0 < \beta < \beta^* \equiv 1{\cdot}414\ldots$)

Here, both (5) and (11), subject to (7) and (13), respectively, admit symmetric solutions, hence the solution to (11) and (13) can be constructed simply by means of a regular perturbation expansion. Since $F < 1$, we expand η and q in a power series in F,

$$\eta = \eta_0 + F\eta_1 + F^2\eta_2^2 + \cdots, \tag{18}$$

$$q = q_0 + Fq_1 + F^2 q_2 + \cdots \tag{19}$$

and then substitute these in the modified lubrication Equation (11). The corresponding leading-order equation, $O(1)$, is

$$q_0 = \eta_0 - \frac{\eta_0^3}{3}\cos\theta, \text{ subject to} \tag{20}$$

$$\beta = \frac{1}{\pi}\int_{-\pi}^{\pi} \eta_0 \, \mathrm{d}\theta. \tag{21}$$

Since β is given, we can solve (20) to obtain η_0 which has to satisfy (21). Clearly, the magnitude of η_0 is $O(\beta)$. To next order, $O(F)$, the equation is

$$q_1 = \eta_1\left\{1 - \eta_0^2\cos\theta\right\} + \frac{\eta_0^3}{3\beta}\left\{\eta_0\cos\theta + \eta_0'\sin\theta\right\}, \text{ subject to} \tag{22}$$

$$0 = \int_{-\pi}^{\pi}\eta_1\,\mathrm{d}\theta - \frac{1}{2\beta}\int_{-\pi}^{\pi}\eta_0^2\,\mathrm{d}\theta. \tag{23}$$

Since both η_0 and β are known, we can solve for η_1 which satisfies (23). The other higher-order terms in (18) can be obtained by following a similar procedure. This case will therefore not be pursued any further.

2.2. Case b ($\beta^* < \beta < \beta^{**} \equiv 2{\cdot}21\ldots$)

Here, as was already noted in the introduction, the solution to (5) and (7) is no longer symmetric and has a discontinuity at $\theta = \theta^*$, with the value of θ^*, lying between $-\pi/2$ and 0, being determined by the given value of β. For $\alpha \to 0$ and β fixed, we seek therefore to construct an asymptotic solution to (11) and (13) in which (5), with $q = 2/3$ and subject to (7), serves as the first term of the outer solution valid for $-\pi \le \theta < \theta_-^*$ and $\theta_+^* < \theta \le \pi$. Within the inner region, the last term in (11) clearly plays a significant role and hence, following Johnson [2], we require that the corresponding inner solution $\hat{\eta}$ satisfy:

$$\frac{2}{3} = \hat{\eta} - \frac{1}{3}\hat{\eta}^3\cos\theta^* + \frac{1}{3}\hat{\eta}^3\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}\psi}\sin\theta^* + O(\alpha) \tag{24}$$

with $\psi \equiv (\theta - \theta^*)/\alpha$. As $\psi \to \pm\infty$, the solution of (24) clearly matches with $\eta_+(\theta^*)$ and $\eta_-(\theta^*)$, respectively, the higher and lower positive roots of (5) at $\theta = \theta^*$ and $q = 2/3$. On the other hand, the solution of (24) is not unique given that it is unaffected if an arbitrary constant is added to the definition of ψ. To obtain a unique solution we therefore need to determine the $O(\alpha)$ correction to the outer solution of (11). To this end, we let in the outer region $-\pi \le \theta < \theta_-^*$ and $\theta_+^* < \theta \le \pi$,

$$\eta = \eta_0(\theta) + \alpha\eta_1(\theta) + \cdots, \tag{25}$$

where, as before, $\eta_0(\theta)$ satisfies (5) with $q = 2/3$ and subject to (7). Equation (25), when substituted in (11) and (13) and, in view of the fact that $q = 2/3 + \alpha/3 + \cdots$ (*cf.* [6, top of p.17]) leads to

$$\frac{1}{3} = (1 - \eta_0^2\cos\theta)\eta_1 + \frac{1}{3}(\eta_0^4\cos\theta + \eta_0^3\eta_0'\sin\theta), \tag{26}$$

hence

$$\eta_1 = \frac{1 - \eta_0^4\cos\theta - \eta_0^3\eta_0'\sin\theta}{3(1 - \eta_0^2\cos\theta)} \tag{27}$$

everywhere within the outer region. It can easily be shown that η_1, which equals 5/6 at $\theta=0$, is continuous everywhere within this outer region.

Finally, on applying Equation (13) and taking into account that

$$\beta = \frac{1}{\pi}\int_{-\pi}^{\pi} \eta_0 \, d\theta,$$

we find that

$$0 = \int_{-\pi}^{\theta_-^*}\left(\eta_1 - \frac{1}{2}\eta_0^2\right)d\theta + \int_{\theta_+^*}^{\pi}\left(\eta_1 - \frac{1}{2}\eta_0^2\right)d\theta + \int_{-\infty}^{0}[\hat{\eta} - \eta_-(\theta^*)]d\psi - \int_0^{\infty}[\eta_+(\theta^*) - \hat{\eta}]d\psi, \tag{28}$$

which implicitly determines the unknown constant, for example the value of $\hat{\eta}(0)$, needed for obtaining a unique solution to the inner Equation (24).

Figure 3 shows the film profiles within the boundary layer as obtained from the asymptotic analysis given above and the numerical solution of (11) and (13). Evidently, when $F = 10^{-3}$, there is good agreement between the two sets, thereby leading us to conclude that the boundary-layer-type Equation (24), first presented by Johnson [2], correctly describes the film profile in the intermediate region between the higher and lower positive roots of (5), with $q = 2/3$, near $\theta = \theta^*$, the point of discontinuity of the film profile as obtained by the solution of Equation (5), but only under the asymptotic conditions $\alpha \to 0$ and *fixed* $\beta \equiv F/\alpha$ with $\beta^* < \beta < \beta^{**}$. More specifically, *quantitative* agreement requires that α, or F, be about 10^{-3} or lower.

The analysis presented above presupposes, of course, that surface-tension effects are negligible. This, as mentioned earlier, requires that $\gamma \ll 1$ provided that η' and η''' are both $O(\eta)$. This latter condition is not satisfied within the inner region, however, and, as can easily be seen by comparing (16) and (24), the surface-tension contribution is negligible within this inner region $\theta - \theta^* \sim O(\alpha)$ only if $\gamma \ll \alpha^2$, a requirement which, of course, is much more restrictive than $\gamma \ll 1$. On the other hand, if $\gamma > O(\alpha^2)$ but still much less than unity, surface-tension effects will alter the film profile only within the inner region, leaving the outer solution essentially unchanged.

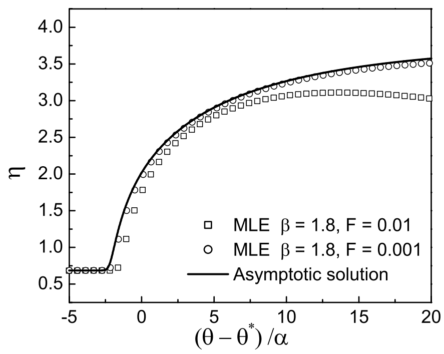

Figure 3. The thickness profile as obtained from the asymptotic solution of Equation (24) is compared with those computed from the numerical solution of the modified lubrication equation (MLE) for $\beta = 1{\cdot}8 < \beta^{**}$.

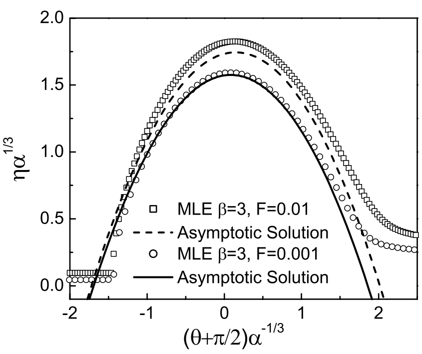

Figure 4. Two film thickness profiles as obtained from the asymptotic solution of Equation (36) are compared with those computed from the numerical solution of the modified lubrication equation (MLE) for $\beta = 3 > \beta^{**}$.

2.3. Case c ($\beta > \beta^{**} \equiv 2{\cdot}21\ldots$)

In this case, Equation (5) subject to (7), does not have a solution and therefore the construction of the asymptotic expansion has to proceed with care. First of all, we note that, as $\beta \to \beta^{**}$ from below, the dimensionless film thickness becomes unbounded but integrable at $\theta = -\pi/2$, the point of symmetry of the liquid puddle that forms when the cylinder stops rotating ($\alpha = 0$). Therefore, with $\beta_1 = \beta - \beta^{**} > 0$, we seek an asymptotic solution of (11) subject to (13) in which the solution of the algebraic equation (5) with $q = 2/3$ and $\beta_1 = 0$ serves as the outer solution on either side of the point of discontinuity $\theta = -\pi/2$, plus an asymptotically thin boundary layer within which η increases without bound as $\alpha \to 0$. To this end, let

$$\Phi \equiv \theta + \pi/2$$

in terms of which Equation (11) becomes,

$$q = \eta - \frac{1}{3}\eta^3\left\{(1 - \alpha\eta)\sin\Phi + \alpha\eta'\cos\Phi\right\}. \tag{29}$$

Letting $\alpha \to 0$, we therefore recover the outer equation

$$\frac{2}{3} = \eta^* - \frac{1}{3}(\eta^*)^3 \sin\Phi$$

for

$$\Phi > 0_+ \text{ and } \Phi < 0_-,$$

for which

$$\frac{1}{\pi}\int_{-\pi/2}^{3\pi/2} \eta^*\,\mathrm{d}\Phi = \beta^{**} \equiv 2{\cdot}21\ldots.$$

We note that, as $\Phi \to 0_+$, $\eta^* \to (3/\Phi)^{1/2}$, and that $\eta^* \to 2/3$ as $\Phi \to 0_-$.

Next, we seek a transformation $\hat{\eta} = \eta\alpha^a$, $\hat{\Phi} = \Phi\alpha^{-b}$, with a and b both positive, such that $\hat{\eta}$ and $\hat{\Phi}$ become $O(1)$ within the boundary layer. In view of (29), we are left with two choices:
(i) $a = 1/5$, $b = 2/5$ in which case (29) becomes

$$\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}\hat{\Phi}} = -\hat{\Phi} + \frac{3}{\hat{\eta}^2} + O(\alpha^{1/5}) \tag{30}$$

or
(ii) $a = b = 1/3$

$$\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}\hat{\Phi}} = -\hat{\Phi} + O(\alpha^{1/3}). \tag{31}$$

The first choice, however, can increase the value of

$$\beta_1 \equiv \beta - \beta^{**} = \frac{1}{\pi}\int_{-\pi/2}^{3\pi/2} (\eta - \eta^*)\mathrm{d}\Phi \tag{32}$$

by an amount of at most $O(\alpha^{1/5})$ and, hence, cannot lead to a solution for $\beta_1 = O(1)$. We are left then with the second possibility, Equation (31), the solution to which is

$$\hat{\eta} = A\left\{1 - \frac{\hat{\Phi}^2}{2A}\right\}, \tag{33}$$

where, in view of (32),

$$A = \frac{1}{2}\left(\frac{3\pi\beta_1}{2}\right)^{2/3} \tag{34}$$

with an error of $O(\alpha^{1/6})$, as can be shown by straightforward analysis involving matching between (33), as $\hat{\Phi} \to \pm\sqrt{2A}$, and the outer solution η^* as $\Phi \to 0$.

Comparing (11) with (5), we find that the maximum value of η, as obtained from the solution of the MLE (11) when $\beta \geq \beta^{**}$, lies essentially on the η *vs.* θ curve found from the solution of the standard lubrication equation (SLE) (5) when $\beta = \beta^{**}$ according to which, as mentioned earlier, $\eta = \left(\frac{3}{\pi/2+\theta}\right)^{1/2} + O(1)$ at $\theta = -\pi/2 + \varepsilon$ with $\varepsilon \to 0$. Therefore, we should center the inner solution, not at $\theta = -\pi/2$ but at $\theta = -\pi/2 + \varepsilon$. Rescaling the variables, η and θ, and comparing with the maximum value of the inner solution (33), we obtain

$$\varepsilon = \frac{3}{A^2}\alpha^{2/3}, \tag{35}$$

provided that $\varepsilon \to 0$. Therefore, the inner solution becomes,

$$\hat{\eta} = A\left\{1 - \frac{(\hat{\Phi} - \varepsilon)^2}{2A}\right\}. \tag{36}$$

We remark parenthetically that, as can be seen from the analysis given above plus (16), the capillary terms remain negligible within this boundary layer provided that $\gamma \ll \alpha^{2/3}$. This is less stringent than the corresponding condition ($\gamma \ll \alpha^2$) which applies within the boundary layer of Case b. But, even if $\gamma > O(\alpha^{2/3})$, surface-tension effects will alter the film profile only within the inner solution and leave the outer solution essentially unchanged, if $\gamma \ll 1$, as was the case in (b).

The foregoing analysis shows then that, as $\alpha \to 0$, with *fixed* $\beta_1 = \beta - \beta^{**} > 0$, the film profile is given by: the discontinuous solution of (5) with $q = 2/3$ (and with the discontinuity located at $\theta = -\pi/2$ where the largest of the two real roots of (5) is infinite), plus a narrow but relatively deep puddle symmetrically placed on either side of $\theta = -\pi/2$ within which the extra fluid given by the positive value β_1 is accumulated. Again, as was the case where $\beta^* < \beta < \beta^{**}$, there is a good agreement between the numerical solution of (11) and the asymptotic solution developed above for $\beta = 3$, as can be seen in Figure 4. It is also clear, however, that the asymptotic analysis applies only for exceedingly small values of α (or for F when β is fixed). For example, for $\beta = 3$ ($\beta_1 = 0{\cdot}79\ldots$), the asymptotic solution developed above is in good *quantitative* agreement with the numerical solution of (11) and (13) only if α or F are below, approximately, 10^{-3}.

We can conclude, therefore, that for *fixed* $\beta > 1{\cdot}4142\ldots$ and $\alpha \to 0$, the solution to (5) and (7), together with the asymptotic analysis just presented, represents the salient features of the liquid-film profile but under extremely restrictive conditions which typically require very small values of the fill fraction F (or of α). In contrast, as shown in [6], the solutions of (5) subject to (7) and those of (11) subject to (13), have little in common if F is held fixed and, as α decreases towards zero, β exceeds $\beta^* \equiv 1{\cdot}4142\ldots$.

3. Stability of the two-dimensional steady solutions to two-dimensional disturbances

This issue has already been examined in [3] on the basis of the standard lubrication Equation (5) subject to (7), where it was shown that: (a) the symmetric solutions ($\beta < \beta^*$) are neutrally stable; and (b) the asymmetric discontinuous solutions ($\beta^* < \beta < \beta^{**}$) are asymptotically stable. We wish to examine whether these conclusions still apply if the base flow is given by the solutions of the MLE, Equations (11) or (16) subject to (13) which are everywhere continuous over the whole range of $0 < \beta < \infty$. Note, however, that, as shown previously [7], β^* is no longer a constant is this case, but rather varies slightly with α.

Following [3], we therefore let $q = q_0 + \delta$, $\eta = \eta_0 + \xi$ with $\xi \ll \eta_0$ everywhere (the subscript 0 refers to the unperturbed steady state) and then linearize Equations (11) and (13) to yield

$$\delta = f_0(\theta)\xi + \frac{\alpha}{3}\eta_0^3 \frac{\partial \xi}{\partial \theta}\sin\theta, \tag{37}$$

where

$$f_0(\theta) \equiv 1 - \eta_0^2 \cos\theta + \frac{4\alpha}{3}\eta_0^3 \cos\theta + \alpha\eta_0^2\eta_0' \sin\theta = \frac{3q_0}{\eta_0} - 2 + \frac{\alpha}{3}\eta_0^3 \cos\theta. \tag{38}$$

On substituting the above in the kinematic condition

$$(1 - \alpha\eta_0)\frac{\partial \xi}{\partial t} + \frac{\partial q}{\partial \theta} = 0 \tag{39}$$

and then separating variables by letting

$$\xi = \exp(\lambda t) G(\theta), \tag{40}$$

we therefore obtain, for the eigenvalue λ and eigenfunction $G(\theta)$,

$$\lambda(1 - \alpha\eta_0)G + \frac{d}{d\theta}(f_0 G) + \frac{\alpha}{3}\frac{d}{d\theta}\left[\eta_0^3 \frac{dG}{d\theta}\sin\theta\right] = 0, \tag{41}$$

which must be solved subject to the periodicity and integral constraints respectively, *i.e.*,

$$G(\theta + 2\pi) = G(\theta) \quad \text{and} \quad \int_{-\pi}^{\pi} (1 - \alpha\eta_0) G \, d\theta = 0. \tag{42}$$

We shall consider separately the case of (a) homogeneous profiles ($\beta < \beta^*$) for which f_0 and η_0 are even functions of θ; and (b) inhomogeneous profiles with $\beta > \beta^*$.

3.1. (a) Stability of the homogeneous film ($\beta < \beta^*$)

Following BPT [3], we let λ^* and G^* be, respectively, the complex conjugate of the indicated complex quantity. Then, on multiplying (41) by $f_0 G^*$ and its conjugate by $f_0 G$, adding the two equations followed by integrating this sum from $-\pi$ to π, we obtain after straightforward manipulations that

$$(\lambda + \lambda^*) \int_{-\pi}^{\pi} (1 - \alpha\eta_0) f_0 |G|^2 \, d\theta = \frac{\alpha}{3} \int_{-\pi}^{\pi} \eta_0^3 \sin\theta \left\{ 2 f_0 \left| \frac{dG}{d\theta} \right|^2 + \frac{d f_0}{d\theta} \frac{d|G|^2}{d\theta} \right\} d\theta. \tag{43}$$

Thus, since the integral on the left-hand side of (43) is positive, given that f_0 is everywhere positive, we conclude that λ is complex if $|G|^2$ and $|dG/d\theta|^2$ are even functions of θ, given that f_0 and η_0 are also even functions. To see whether this is so, we seek to construct a solution of (41) and (42) by successive approximations in which, to begin with, we neglect the last term in (41) [This is permissible, given that $\eta_0 \sim O(\beta)$ when $\beta < \beta^*$, hence $\alpha\eta_0^3/3 = O(F\beta^2/3)$ and therefore small for $F \leq 0{\cdot}36$]. Consequently, to a first approximation,

$$\lambda(1 - \alpha\eta_0)G + \frac{d}{d\theta}(f_0 G) \cong 0, \tag{44}$$

the solution to which, periodic in 2π, is

$$G=\frac{c}{f_0}\exp\left\{-\lambda\int_0^\theta\frac{1-\alpha\eta_0}{f_0}\,\mathrm{d}\theta\right\}, \tag{45}$$

also given by O'Brien [10], where c is an arbitrary constant (to be set, in most cases and without loss of generality, equal to unity), and

$$\lambda=2n\pi\mathrm{i}/A \quad\text{with}\quad A=\int_{-\pi}^{\pi}\frac{1-\alpha\eta_0}{f_0}\,\mathrm{d}\theta=2\int_0^{\pi}\frac{1-\alpha\eta_0}{f_0}\,\mathrm{d}\theta,$$

n being a positive integer. [Note that the eigenfunction $G=c/f_0$ corresponding to $n=0$, does not satisfy the integral constraint given by (42).] Clearly, to a first approximation, the homogeneous film with symmetric profiles is neutrally stable as was already shown in [3] using, as the base state, that given by the solution of (5) and (7). Furthermore, we note that the real part of G, as given by (45), is even and the imaginary part is odd, hence, as expected from the symmetry of the base film-thickness profile η_0, both $|G|^2$ and $|\mathrm{d}G/\mathrm{d}\theta|^2$ are symmetric and, the right-hand side of (43) vanishes identically. Therefore, the symmetric steady solution remains neutrally stable, even if the analysis is carried out to $O(\alpha)$.

To examine whether this is still the case if the expansion is continued to higher order in α, we formally solve (41) and find, in addition to the homogeneous term (45), the particular solution G_p due to the last term in (41)

$$\begin{aligned}\frac{3f_0G_p}{\alpha}=&-\eta_0^3\sin\theta\frac{\mathrm{d}G}{\mathrm{d}\theta}+\lambda\exp\left\{-\lambda\int_0^\theta\frac{1-\alpha\eta_0}{f_0}\mathrm{d}\theta\right\}\\&\times\int_0^\theta\frac{1-\alpha\eta_0}{f_0}\exp\left\{\lambda\int_0^\theta\frac{1-\alpha\eta_0}{f_0}\mathrm{d}x\right\}\eta_0^3\frac{\mathrm{d}G}{\mathrm{d}\theta}\sin\theta\mathrm{d}\theta,\end{aligned} \tag{46}$$

where λ is given by the corresponding expression in (45) to insure that G_p remains periodic in 2π. On substituting the expression for G given in (45) in the right-hand side of (46), we can easily show that, to $O(\alpha)$, $|G|^2$ and $|\mathrm{d}G/\mathrm{d}\theta|^2$ remain even, hence, as expected, the right-hand side of (43) vanishes to $O(\alpha^2)$ and λ remains purely imaginary to this order. In fact, on using successive approximations, it is not difficult to see from (45) that the conclusions given above remain valid to all orders in α. Thus, the inclusion of the hydrostatic-pressure term which led to the improved version of the lubrication equations, *viz.* Equations (11) and (13), has not led to a corresponding improvement in the stability characteristics of the symmetric film profiles which, according to the foregoing temporal stability analysis, remain neutrally stable. This conclusion regarding the neutral stability of the homogeneous films runs counter to the results of a recent analysis [11,12] according to which the high-frequency ($n\to\infty$) eigenmodes of (44) (with $\alpha=0$), if added together, can generate an "explosive" instability of these films. Unfortunately, as was already noted [12] and as is obvious from the basic assumptions underlying the lubrication analysis, such high-frequency modes are incompatible with (44) being a good first approximation to (41), given that derivates with respect to θ are no longer $O(1)$. Thus, given that there exists a cut-off frequency above which these high-frequency modes will have to satisfy an equation substantially more complicated than (44), the physical existence of such "explosive" instabilities is open to question.

In order to arrive at a definite conclusion regarding the stability of the homogeneous film, we therefore turn to the small capillary effects, *i.e.*, the last two terms in (16) which we have neglected thus far in our analysis. Therefore, in lieu of (37), we have

$$\delta=f_0(\theta)\xi+\frac{\alpha}{3}\eta_0^3\frac{\partial\xi}{\partial\theta}\sin\theta+\frac{\alpha}{3}\gamma\eta_0^3\left(\frac{\partial\xi}{\partial\theta}+\frac{\partial^3\xi}{\partial\theta^3}\right), \tag{47}$$

where (*cf.* (38))

$$f_0(\theta) \equiv 1-\eta_0^2\cos\theta+\frac{4\alpha}{3}\eta_0^3\cos\theta+\alpha\eta_0^2\eta_0'\sin\theta+\alpha\gamma\eta_0^2(\eta_0'+\eta_0''')$$
$$=\frac{3q_0}{\eta_0}-2+\frac{\alpha}{3}\eta_0^3\cos\theta \tag{48}$$

with η_0 being the solution of (16) subject to (13) for given q_0. Note that the homogeneous film in this case is no longer symmetric about $\theta=0$.

By repeating the arguments used previously, we therefore arrive at the equation for the eigenvalue λ and the eigenfunction G,

$$\lambda G+\frac{\mathrm{d}}{\mathrm{d}\theta}(f_0G)+\frac{\alpha}{3}\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\eta_0^3\frac{\mathrm{d}G}{\mathrm{d}\theta}\sin\theta\right]+\frac{\alpha}{3}\gamma\frac{\mathrm{d}}{\mathrm{d}\theta}\left\{\eta_0^3\left(\frac{\mathrm{d}G}{\mathrm{d}\theta}+\frac{\mathrm{d}^3G}{\mathrm{d}\theta^3}\right)\right\}=0, \tag{49}$$

hence, in lieu of (43), we have

$$(\lambda+\lambda^*)\int_{-\pi}^{\pi}f_0|G|^2\mathrm{d}\theta=\frac{\alpha}{3}\int_{-\pi}^{\pi}\eta_0^3\sin\theta\left\{2f_0\left|\frac{\mathrm{d}G}{\mathrm{d}\theta}\right|^2+\frac{\mathrm{d}f_0}{\mathrm{d}\theta}\frac{\mathrm{d}|G|^2}{\mathrm{d}\theta}\right\}\mathrm{d}\theta$$
$$+\frac{\alpha}{3}\gamma\int_{-\pi}^{\pi}\eta_0^3\left[2f_0\left|\frac{\mathrm{d}G}{\mathrm{d}\theta}\right|^2+\frac{\mathrm{d}f_0}{\mathrm{d}\theta}\frac{\mathrm{d}|G|^2}{\mathrm{d}\theta}\right.$$
$$\left.+\frac{\mathrm{d}(f_0G)}{\mathrm{d}\theta}\frac{\mathrm{d}^3G^*}{\mathrm{d}\theta^3}+\frac{\mathrm{d}(f_0G^*)}{\mathrm{d}\theta}\frac{\mathrm{d}^3G}{\mathrm{d}\theta^3}\right]\mathrm{d}\theta. \tag{50}$$

As discussed earlier, it is permissible, to a first approximation, to neglect the last two terms in (49), given that both $\alpha\eta_0^3/3$ as well as γ are small, so that we have

$$G=\frac{1}{f_0}\exp\left\{-\frac{2n\pi\mathrm{i}}{A}\int_0^{\theta}\frac{\mathrm{d}\theta}{f_0}\right\}\quad\text{with}\quad A\equiv\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta}{f_0}. \tag{51}$$

On using the above expression for G, we can show, after some algebra, that the first and second terms on the right-hand side of (50) reduce, respectively to

$$\frac{2\alpha}{3}\left(\frac{2\pi n}{A}\right)^2\int_{-\pi}^{\pi}\frac{\eta_0^3}{f_0^3}\sin\theta\,\mathrm{d}\theta \tag{52}$$

and

$$-\frac{4\alpha\gamma}{3}\left(\frac{2\pi n}{A}\right)^2\int_{-\pi}^{\pi}\frac{\eta_0^3}{2f_0^5}\left\{\left(\frac{2\pi n}{A}\right)^2-f_0^2-11\left(\frac{\mathrm{d}f_0}{\mathrm{d}\theta}\right)^2+4f_0\frac{\mathrm{d}^2f_0}{\mathrm{d}\theta^2}\right\}\mathrm{d}\theta. \tag{53}$$

To evaluate (52) we note that, to a first approximation from (48), $f_0=1-\eta_0^2\cos\theta$, with η_0 given by the SLE (5) subject to (7), so that, to leading order, the integral in (52) vanishes. Continuing to the next order, we find that the leading-order contribution to (52) is $O(\alpha^2\gamma)$ which is smaller than $O(\alpha\gamma)$, the leading-order term of (53). We therefore focus on (53), setting $f_0=1-\eta_0^2\cos\theta$ as a first approximation and find by numerical integration that the whole expression given in (53) is negative for all $0<\beta<\beta^*$. A plot *vs.* β of the (positive) integral in (53) for $n=1$, which is obviously the least stable mode, is shown in Figure 5. It should be noted that this conclusion regarding the (negative) sign of the right-hand of (50) is only a first approximation which, however, should become increasingly more accurate as $F\to 0$ for β fixed.

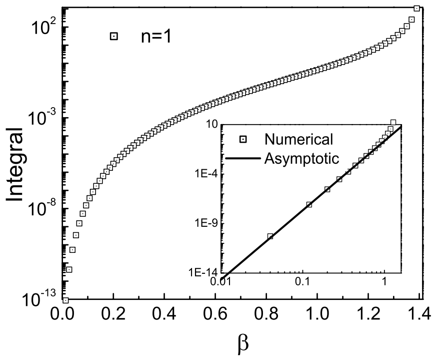

Figure 5. The values of the integral in (53) for $n=1$ in the range of $0<\beta<\beta^*\equiv 1{\cdot}414\ldots$. In the small window, for small β, the values of the integral given by the asymptotic expression (55) are in good agreement with these obtained by numerical integration.

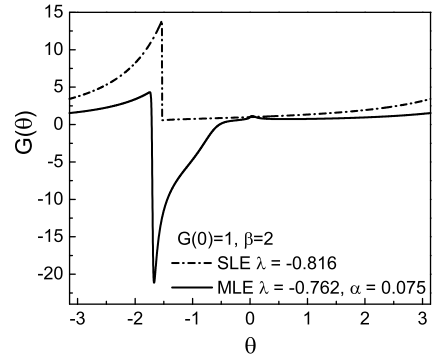

Figure 6. The eigenfunction $G(\theta)$, with $G(0)=1$, as obtained from the modified lubrication (MLE) and from the standard lubrication equation (SLE) for $\beta=2{\cdot}0$ and $\alpha=0{\cdot}075$. Note that the former is continuous whereas the latter is discontinuous.

In addition, the asymptotic expression for the real part of the eigenvalue λ (*i.e.*, $(\lambda+\lambda^*)/2$) as $\beta\to 0$ can be derived as follows:

From the standard lubrication Equation (5) subject to (7), we find that,

$$\eta_0=\frac{\beta}{2}+\frac{1}{3}\left(\frac{\beta}{2}\right)^3\cos\theta+\text{h.o.t. in }\beta, \tag{54}$$

provided that $\beta\ll 1$. On substituting the above in the integral in (53), we can show, after some algebra, that, for $n=1$, the leading-order term of the integral is of the order of $O(\beta^7)$ with its coefficient being $9\pi/2^7$. As shown in Figure 5, there is excellent agreement for small β between the values of the integral as obtained by numerical integration and the asymptotic expression quoted above. Therefore, according to (50) and (53), we have,

$$\frac{\lambda+\lambda^*}{2}=-3\alpha\gamma\left(\frac{\beta}{2}\right)^7+\text{h.o.t. in }\beta. \tag{55}$$

This means that, as $\beta\to 0$, the two-dimensional disturbances to the two-dimensional steady-state film decay according to $\exp(-3\alpha\gamma(\beta/2)^7 t)$, in complete agreement with the results given recently by Hinch and Kelmanson [13]. We see, therefore, that, according to the foregoing analysis, the homogeneous film is indeed asymptotically stable to small disturbances, but only weakly so, due to the weak capillary forces which have essentially no effect on the structure of the basic flow whenever $\alpha\gamma\ll 1$. It is surprising that, as $\beta\to 0$, the thin film becomes less stable, given that the decay rate of the small disturbances is only $O(\beta^7)$. The physical mechanisms governing such a slow decay involving a delicate interplay between rotation, gravity and surface tension was further discussed in [13].

3.2. (b) Stability of the inhomogeneous film ($\beta>\beta^*$)

The equation for the eigenvalue λ and the eigenfunction $G(\theta)$ is the same as that in the previous case, *viz*. Equation (41) for the homogeneous film. But, before solving this equation, we shall study the equation for the eigenvalue λ and the eigenfunction G corresponding to the standard lubrication equation, *viz*.

$$\lambda G+\frac{\mathrm{d}(f_0 G)}{\mathrm{d}\theta}=0 \quad \text{with} \quad f_0=1-\eta_0^2\cos\theta, \tag{56}$$

which applies everywhere, except at the point of discontinuity $\theta = \theta^*$. The eigenfunction G has to satisfy the periodic boundary condition.

To obtain the eigenvalue λ in (56), we expand both $\eta_0(\theta)$ and the eigenfunction $G(\theta)$ in a power series about the midpoint ($\theta = 0$),

$$\eta_0(\theta) = \sum_0^\infty a_k \theta^k \quad \text{and} \quad G(\theta) = \sum_0^\infty b_k \theta^k$$

and substitute these in (56). The leading-order term ($O(1)$) is

$$(\lambda - 2a_0 a_1) b_0 + (1 - a_0^2) b_1 = 0. \tag{57}$$

But since $a_0 = 1$ and $a_1 = -1/\sqrt{6}$ when $\beta > \beta^* \equiv 1{\cdot}414\ldots$, as can be seen by substituting the power series of η in the standard lubrication Equation (5) with $q = 2/3$, the second term in (57) vanishes automatically. Thus we obtain one eigenvalue

$$\lambda = 2a_0 a_1 = -\sqrt{\frac{2}{3}}. \tag{58}$$

In order to obtain all the other eigenvalues, we expand $G(\theta)$ in the general form,

$$G(\theta) = \theta^{n-1} \sum_0^\infty b_k \theta^k,$$

where n is a positive integer, and substitute this together with the expansion of η used previously in (56). The two leading-order terms ($O(1)$) and ($O(\theta)$) are

$$(n-1)(1 - a_0^2) b_0 = 0 \tag{59}$$

and

$$(\lambda - 2n a_0 a_1) b_0 + n(1 - a_0^2) b_1 = 0, \tag{60}$$

respectively. But, since $a_0 = 1$, Equation (59) is automatically satisfied, while, from (60), we obtain all the eigenvalues

$$\lambda = 2n a_0 a_1 = -\sqrt{\frac{2}{3}} n \tag{61}$$

with $n = 1$ clearly giving the maximum eigenvalue, $\lambda = -\sqrt{\frac{2}{3}}$, which we have already obtained (*cf.* (58)). It may seem surprising that all these eigenvalues can be obtained simply via a local expansion of the corresponding eigenequation (here, about $\theta = 0$), rather than from a complete solution of this equation as is usually the case, but there exists at least one well-known example where this is also the case, *viz*. Legendre's equation where the (integer) eigenvalues are obtained by requiring that the eigensolutions should remain regular at the singular points of the equation.

The general form of the corresponding eigenfunction G for $n = 1$ is

$$G = \frac{c}{f_0} \exp\left\{ -\sqrt{\frac{2}{3}} \int_\theta^\pi \frac{\mathrm{d}t}{f_0} \right\}, \tag{62}$$

where c is an arbitrary constant. But, since $f_0 \to -2a_0 a_1 \theta = \sqrt{\frac{2}{3}}\theta$ as $\theta \to 0$, Equation (62) needs to be rearranged so it can be applied for all θ.

First, we suppose that $\theta > 0$ and split the integral in (62) into two parts,

$$\int_\theta^\pi \frac{\mathrm{d}t}{f_0} = \int_\theta^\pi \sqrt{\frac{3}{2}}\frac{\mathrm{d}t}{t} + \int_\theta^\pi \left\{\frac{1}{f_0} - \sqrt{\frac{3}{2}}\frac{1}{t}\right\}\mathrm{d}t,$$

where the second term on the right-hand side becomes $O(1)$ as $\theta \to 0$. Substituting this in (62), we obtain

$$G = \frac{c}{\pi}\frac{\theta}{f_0}\exp\left\{-\sqrt{\frac{2}{3}}\int_\theta^\pi \left(\frac{1}{f_0} - \sqrt{\frac{3}{2}}\frac{1}{t}\right)\mathrm{d}t\right\}, \tag{63}$$

which is well-behaved around $\theta \to 0$ and can be shown to satisfy (56) with $\lambda = -\sqrt{\frac{2}{3}}$, even for $\theta < 0$ up to the discontinuity at $\theta = \theta^*$.

Now, let $G(0) = 1$ without loss of generality, hence

$$G = \sqrt{\frac{3}{2}}\frac{\theta}{f_0}\exp\left\{\sqrt{\frac{2}{3}}\int_0^\theta \left(\frac{1}{f_0} - \sqrt{\frac{3}{2}}\frac{1}{t}\right)\mathrm{d}t\right\}, \tag{64}$$

given that, as shown above, $f_0 \to \sqrt{\frac{2}{3}}\theta$ as $\theta \to 0$. However, Equation (64) only applies for $\theta^* < \theta \le \pi$. On the other hand, according to (62), the eigenfunction G with $G(0) = 1$ for $-\pi \le \theta < \theta^*$ is,

$$G = \frac{c}{f_0}\exp\left\{\sqrt{\frac{2}{3}}\int_{-\pi}^\theta \frac{\mathrm{d}t}{f_0}\right\},$$

where $c = f_0(\pi)G(\pi)$ given that $G(-\pi) = G(\pi)$ and $f_0(-\pi) = f_0(\pi)$. Note that $G(\pi)$ can be obtained from (64). In addition, on account of f being discontinuous, G is also discontinuous at $\theta = \theta^*$ and, due to the existence of such a discontinuity, $f_0(\theta^*_-)G(\theta^*_-) \neq f_0(\theta^*_+)G(\theta^*_+)$. Although such a discontinuity in $f_0 G$ may seem surprising at first glance in view of Equation (56), it has been shown [3] that, at $\theta = \theta^*$, this equation contains an extra term involving a delta function the integral of which exactly cancels the discontinuity in $f_0 G$ referred to above.

Now, returning to (41), we substitute the power series expansions of η and of G, given previously, in (41). When $n = 1$, the leading-order term ($O(1)$) gives

$$\left\{\lambda(1-\alpha a_0) - 2a_0 a_1\left(1 - \frac{5}{2}\alpha a_0\right)\right\} b_0 + \left\{1 - a_0^2 + \frac{5}{3}\alpha a_0^3\right\} b_1 = 0. \tag{65}$$

But since, for an asymmetric base-state film profile $\left\{1 - a_0^2 + \frac{5}{3}\alpha a_0^3\right\} = 0$ (*cf.* [6], Equation 1.7)] we obtain one eigenvalue

$$\lambda = 2a_0 a_1 \frac{1 - \frac{5}{2}\alpha a_0}{1 - \alpha a_0} \tag{66}$$

without solving the whole equation (41). When $n \ge 2$, the $O(1)$ term in the expansion of $G(\theta)$ leads to

$$(n-1)\left\{1 - a_0^2 + \left(1 + \frac{n}{3}\right)\alpha a_0^3\right\} b_0 = 0, \tag{67}$$

while the $O(\theta)$ term gives

$$\left\{\lambda(1-\alpha a_0) - 2a_0 a_1\left[n - \frac{\alpha}{2}(n-1) - \frac{\alpha}{2}a_0(n^2+3n+1)\right]\right\} b_0 + \left\{1 - a_0^2 + (4+n)\frac{\alpha}{3}a_0^3\right\} b_1 = 0, \tag{68}$$

respectively. But, given that $\left\{1-a_0^2+\frac{5}{3}\alpha a_0^3\right\}=0$, we find that, although (67) is automatically satisfied only when $n=2$, the second term in (68) does not vanish automatically for any $n\geq 2$. Hence, using this method, we can only obtain one eigenvalue ($n=1$), *viz.* (66). But, in view of our findings regarding the eigenvalues given by the standard lubrication equation, *viz.* (58), we can safely take it for granted that the eigenvalue given by (66) corresponds to the least stable mode and that the remaining eigenvalues, which can only be obtained via a numerical solution of the full eigenequation, can be ignored. Moreover, given that a_0 and a_1 are only weakly dependent on β for $\beta>\beta^*$ and fixed F, it is evident that λ, as given by (66), is similarly essentially constant – a somewhat surprising result. Moreover, in view of [6], $a_0=1+5\alpha/6+O(\alpha^2)$ and $a_1=-1/\sqrt{6}+O(\alpha)$, we have that

$$\lambda=2a_0a_1\frac{1-\frac{5}{2}\alpha a_0}{1-\alpha a_0}\cong-\sqrt{\frac{2}{3}}+O(\alpha)<0. \tag{69}$$

Hence, in agreement with the result found previously [3,10] using the standard lubrication analysis (SLA), the stability analysis given above shows that the liquid film is indeed asymptotically stable when the film profile becomes asymmetric, even for zero surface tension. Note that, when $\alpha\to 0$ in (69), the eigenvalue λ becomes equal to $-\sqrt{\frac{2}{3}}$, which is precisely the maximum eigenvalue given by the standard lubrication analysis, even though the latter no longer applies when $\beta>\beta^{**}$. This, indeed, is a very curious result. Shown in Figure 6, for a particular case, is a plot of the eigenfunction $G(\theta)$ *vs.* θ showing that the former is everywhere continuous if we use the modified lubrication Equations (11) and (13). The corresponding eigenfunction, given by (62), is also plotted. Clearly, the two eigenfunctions are very different, even though the corresponding eigenvalues differ by only approximately 7% in this case.

4. Stability of the two-dimensional steady solutions to three-dimensional disturbances

This issue also has been examined in [3], on the basis of the standard lubrication equation (5) subject to (7), where it was shown that the steady flow is neutrally stable and that steady perturbations exist for arbitrary wavelength along the cylinder axis as long as the liquid-film profile is symmetric ($\beta<\beta^*$); on the other hand, when the liquid profile is asymmetric ($\beta^*<\beta<\beta^{**}$), the steady solutions are asymptotically stable. Here, using the continuous solutions of the modified lubrication Equations (11) or (16) subject to (13), we wish to examine whether or not these conclusions remain valid for the whole range of $0<\beta<\infty$. Note that, as remarked earlier, β^* given by the MLE is no longer a constant but rather varies slightly with α.

In the thin-film approximation, the dimensionless axial velocity $w(\equiv\hat{w}/\Omega R$ with $\hat{w}$ being the dimensional one) is given by [3,7,14]

$$w=\alpha\left(\eta z-\frac{1}{2}z^2\right)\frac{\partial\eta}{\partial x}\sin\theta, \tag{70}$$

where $x\equiv\hat{x}/R$ is the dimensionless axial coordinate with $\hat{x}$ being the dimensional one. Consequently, the volume-conservation equation for a three-dimensional flow is [3,7,14]

$$(1-\alpha\eta)\frac{\partial\eta}{\partial t}+\frac{\partial q}{\partial\theta}+\frac{\alpha}{3}\frac{\partial}{\partial x}\left(\eta^3\frac{\partial\eta}{\partial x}\right)\sin\theta=0. \tag{71}$$

This equation can also be derived starting from the MLA. Letting $\eta = \eta_0 + \xi$ with $\xi \ll \eta_0$ and in view of (37) and (38), we therefore arrive at

$$(1-\alpha\eta_0)\frac{\partial \xi}{\partial t} + \frac{\partial (f_0 \xi)}{\partial \theta} + \frac{\alpha}{3}\frac{\partial}{\partial \theta}\left[\eta_0^3 \frac{\partial \xi}{\partial \theta}\sin\theta\right] + \frac{\alpha}{3}\eta_0^3 \frac{\partial^2 \xi}{\partial x^2}\sin\theta = 0 \tag{72}$$

which, on separating variables by letting

$$\xi = \exp(\lambda t) G(\theta) \sin(kx+b), \tag{73}$$

where k and b are real constants, becomes

$$\lambda(1-\alpha\eta_0)G + \frac{\mathrm{d}(f_0 G)}{\mathrm{d}\theta} + \frac{\alpha}{3}\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\eta_0^3 \frac{\mathrm{d}G}{\mathrm{d}\theta}\sin\theta\right] - \frac{\alpha}{3}k^2 \eta_0^3 G \sin\theta = 0. \tag{74}$$

The above must be solved for the eigenvalue λ and eigenfunction G subject to the periodicity,

$$G(\theta + 2\pi) = G(\theta)$$

and the integral constraint,

$$\int_{-(\pi l+b)/k}^{(\pi l+b)/k} \sin(kx+b)\,\mathrm{d}x \int_{-\pi}^{\pi} (1-\alpha\eta_0) G \mathrm{d}\theta = 0 \quad \text{where } l \text{ is integer.} \tag{75}$$

Obviously, the latter constraint is automatically satisfied for any eigenfunction G. In addition, it can easily be shown [15] that the MLA and, therefore, the kinematic condition (71) and the eigenequation (74), are only valid for long-wavelength disturbances, specifically for axial disturbances having wavelengths greater than or comparable to the characteristic thickness of the film.

We shall consider separately the case of (a) homogeneous profiles ($\beta < \beta^*$), and (b) inhomogeneous profiles with $\beta > \beta^*$.

4.1. (a) Stability of the homogeneous film ($\beta < \beta^*$)

By means of the same procedure as used previously in the corresponding two-dimensional case ($\beta < \beta^*$), we begin with a first approximation to Equation (74),

$$\lambda(1-\alpha\eta_0)G + \frac{\mathrm{d}(f_0 G)}{\mathrm{d}\theta} - \frac{\alpha}{3}k^2 \eta_0^3 G \sin\theta \cong 0, \tag{76}$$

the solution to which, periodic in 2π, is

$$G = \frac{1}{f}\exp\left\{\frac{\alpha}{3}k^2 \int_0^\theta \frac{\eta_0^3 \sin\theta}{f_0}\mathrm{d}\theta\right\}\exp\left\{-\lambda \int_0^\theta \frac{1-\alpha\eta_0}{f_0}\mathrm{d}\theta\right\}, \tag{77}$$

where

$$\lambda = 2\pi m i/A \quad \text{with} \quad A = \int_{-\pi}^{\pi} \frac{1-\alpha\eta_0}{f_0}\mathrm{d}\theta = 2\int_0^\pi \frac{1-\alpha\eta_0}{f_0}\mathrm{d}\theta$$

and m is zero or any positive integer. When $m = 0$, there exists a time-independent perturbation with arbitrary dependence on x,

$$\xi = G\sin(kx+b), \tag{78}$$

where

$$G = \frac{1}{f_0}\exp\left\{\frac{\alpha}{3}k^2 \int_0^\theta \frac{\eta_0^3 \sin\theta}{f_0}\mathrm{d}\theta\right\}.$$

Obviously, the above expression for G automatically satisfies periodicity as well as the integral constraint (75), in contrast to the corresponding two-dimensional case, *i.e.*, $G=1/f_0$, which cannot satisfy the integral constraint (42).

Let λ^* and G^* be, respectively, the complex conjugate of the indicated complex quantity. Performing the same operation as shown in the corresponding two-dimensional case, we obtain

$$(\lambda+\lambda^*)\int_{-\pi}^{\pi}(1-\alpha\eta_0)f_0|G|^2\mathrm{d}\theta \\ =\frac{\alpha}{3}\int_{-\pi}^{\pi}\eta_0^3\sin\theta\left\{2f_0\left|\frac{\mathrm{d}G}{\mathrm{d}\theta}\right|^2+\frac{\mathrm{d}f_0}{\mathrm{d}\theta}\frac{\mathrm{d}|G|^2}{\mathrm{d}\theta}+2k^2f_0|G|^2\right\}\mathrm{d}\theta. \tag{79}$$

Since η_0, $|G|^2$ and $|\mathrm{d}G/\mathrm{d}\theta|^2$ are symmetric (this was demonstrated in detail in the previous section dealing with the two-dimensional disturbances), the right-hand side of (79) vanishes. Therefore, the symmetric steady solution remains neutrally stable to the axial disturbances, even if the analysis is carried out to $O(\alpha)$.

Again, as was the case with the stability to two-dimensional disturbances discussed earlier (*cf.* the comments following (46)), the conclusion given above runs counter to the results of a recent analysis [14] according to which, the cumulative effect of short-wavelength axial perturbations (*i.e.*, $k\to\infty$) leads to an "explosive" instability. But since, as noted above, lubrication analysis limits the range of wavenumbers to $k<O(1/\alpha)$, the physical existence of the "explosive" instabilities described in [14], which at these high wavenumbers and frequencies would be anyhow suppressed by surface tension, is also open to question.

Let us next consider the effects of surface tension. Taking account of the capillary contribution arising from the film-thickness variation along axial direction, we extend the expression for the hydrostatic pressure in the two-dimensional case, *i.e.*, Equation (15), to this axially non-uniform case as

$$p=\alpha(z-\eta)\sin(\theta)-\alpha\gamma\left(\eta+\frac{\partial^2\eta}{\partial\theta^2}+\frac{\partial^2\eta}{\partial x^2}\right). \tag{80}$$

Then, by repeating the steps used earlier in the two-dimensional case, we have (*cf.* (47)),

$$\delta=f_0(\theta)\xi+\frac{\alpha}{3}\eta_0^3\frac{\partial\xi}{\partial\theta}\sin\theta+\frac{\alpha}{3}\gamma\eta_0^3\left(\frac{\partial\xi}{\partial\theta}+\frac{\partial^3\xi}{\partial\theta\partial x^2}+\frac{\partial^3\xi}{\partial\theta^3}\right), \tag{81}$$

where f_0 is given by (48) with η_0 being the solution of (16) subject to (13) for given q_0. By repeating the arguments used previously in the two-dimensional case (*cf.* (49)), we arrive at the equation for the eigenvalue λ and the eigenfunction $G(\theta)$,

$$\lambda G+\frac{\mathrm{d}}{\mathrm{d}\theta}(f_0G)+\frac{\alpha}{3}\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\eta_0^3\frac{\mathrm{d}G}{\mathrm{d}\theta}\sin\theta\right]+\frac{\alpha}{3}\gamma\frac{\mathrm{d}}{\mathrm{d}\theta}\left\{\eta_0^3\left[(1-k^2)\frac{\mathrm{d}G}{\mathrm{d}\theta}+\frac{\mathrm{d}^3G}{\mathrm{d}\theta^3}\right]\right\} \\ -\frac{\alpha}{3}k^2\eta_0^3G\sin\theta-\frac{\alpha}{3}\gamma k^2\eta_0^3\left\{(1-k^2)G+\frac{\mathrm{d}^2G}{\mathrm{d}\theta^2}\right\}=0. \tag{82}$$

Note that the last term on the left-hand side of the above equation results from the modification of the axial velocity due to the pressure (80). Hence, in lieu of (79), we have

$$(\lambda+\lambda^*)\int_{-\pi}^{\pi} f_0|G|^2\mathrm{d}\theta=\frac{\alpha}{3}\int_{-\pi}^{\pi}\eta_0^3\sin\theta\left(2f_0\left|\frac{\mathrm{d}G}{\mathrm{d}\theta}\right|^2+\frac{\mathrm{d}f_0}{\mathrm{d}\theta}\frac{\mathrm{d}|G|^2}{\mathrm{d}\theta}+2k^2f_0|G|^2\right)\mathrm{d}\theta$$
$$+\frac{\alpha\gamma k^2}{3}\int_{-\pi}^{\pi}\eta_0^3\left\{2(1-k^2)f_0|G|^2-2f_0\left|\frac{\mathrm{d}G}{\mathrm{d}\theta}\right|^2-\frac{\mathrm{d}f_0}{\mathrm{d}\theta}\frac{\mathrm{d}|G|^2}{\mathrm{d}\theta}+f_0G^*\frac{\mathrm{d}^2G}{\mathrm{d}\theta^2}\right.$$
$$\left.+f_0G\frac{\mathrm{d}^2G^*}{\mathrm{d}\theta^2}\right\}\mathrm{d}\theta+\frac{\alpha\gamma}{3}\int_{-\pi}^{\pi}\eta_0^3\left\{2f_0\left|\frac{\mathrm{d}G}{\mathrm{d}\theta}\right|^2+\frac{\mathrm{d}f_0}{\mathrm{d}\theta}\frac{\mathrm{d}|G|^2}{\mathrm{d}\theta}\right.$$
$$\left.+\frac{\mathrm{d}(f_0G)}{\mathrm{d}\theta}\frac{\mathrm{d}^3G^*}{\mathrm{d}\theta^3}+\frac{\mathrm{d}(f_0G^*)}{\mathrm{d}\theta}\frac{\mathrm{d}^3G}{\mathrm{d}\theta^3}\right\}\mathrm{d}\theta. \tag{83}$$

To evaluate the integrals on the right-hand side of the above equation, we again make use of successive approximations. Clearly, provided that $\alpha\eta_0^3/3$ is small and that $k^2\sim O(1)$ or less, the expression for G in (77) is, to first order,

$$G=\frac{1}{f_0}\exp\left\{-\frac{2m\pi\mathrm{i}}{A}\int_0^\theta\frac{1}{f_0}\mathrm{d}\theta\right\}\quad\text{with}\quad A=\int_{-\pi}^{\pi}\frac{1}{f_0}\mathrm{d}\theta \tag{84}$$

with m being zero or any positive integer, and the leading-order term of f_0 in (48) is

$$f_0=1-\eta_0^2\cos\theta,$$

where η_0 is given by the SLE (5) subject to (7) rather than by (16) subject to (13).

Now, it is obvious that the first term on the right-hand side of (83) vanishes if G and f_0 are given by (84). Then, on repeating the analysis of the two-dimensional case given earlier (*cf.* the comments following (53)), it is easy to show that, to next order, this term becomes $O(\alpha^2\gamma)$ and, therfore is of higher order than the remaining two $O(\alpha\gamma)$ terms on the right-hand of (83). We found, using numerical integration with G and f_0 given by (84), that when m in (84) is any positive integer, the sum of these two terms, is negative for $k>0$ and becomes increasingly more negative with increasing k; however, when $m=0$ (*i.e.*, $G=1/f_0$), the sum has a positive maximum, thereby implying that, according to our linear analysis, the inclusion of the weak capillary effects renders the homogeneous film asymptotically *unstable* to small axial disturbances having a wavelength of the order of the cylinder radius. Such a family of eigenvalue profiles is shown in Figure 7 for three typical cases $\beta=0{\cdot}13$, $0{\cdot}67$ and $1{\cdot}23$. Clearly, the magnitude of the most rapidly growing disturbance for $m=0$ decreases with decreasing β, implying that the symmetric profiles should become more unstable at the larger value of β. It should be noted, however, that, although our foregoing conclusion regarding on the film stability is only a first-order approximation in which the base flow is given by the standard lubrication analysis (SLA), such a conclusion should become more and more accurate as $F\to 0$ for β fixed which is the prerequisite for using the SLA.

In addition, by means of a method similar to that described in the corresponding two-dimensional case (*cf.* (54)), we can also obtain, for $\beta\to 0$, the asymptotic expression for the real part of the eigenvalue λ for $m=0$, *i.e.*,

$$\frac{\lambda+\lambda^*}{2}=\frac{\alpha\gamma}{3}k^2(1-k^2)\left(\frac{\beta}{2}\right)^3+\text{h.o.t in }\beta, \tag{85}$$

which, as shown in Figure 7, closely matches the value of $(\lambda+\lambda^*)/2$ calculated numerically from (83) when β is less than about $0{\cdot}7$. In fact, when $m=0$, the eigenvalue λ is real and

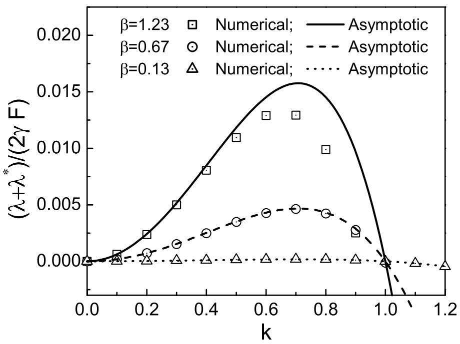

Figure 7. The real part of the eigenvalue λ has a positive maximum when $m=0$ in (84) with the magnitude of the most rapidly amplified disturbance increasing with an increase in the value of β. As $\beta\to 0$, the values of the real part of λ for $m=0$ predicted by the asymptotic expression in Equation (85) are in close agreement with those obtained by numerical calculation from Equation (83) with G given by (84).

so is the corresponding eigenfunction G. Therefore, as $\beta\to 0$, we can expand λ and G in a power series in β, *i.e.*,

$$\lambda = 0 + \lambda^a\left(\frac{\beta}{2}\right)^3 + \lambda^b\left(\frac{\beta}{2}\right)^4 + \text{h.o.t in } \beta$$

and

$$G = \frac{1}{f_0} + G^a\left(\frac{\beta}{2}\right)^3 + G^b\left(\frac{\beta}{2}\right)^4 + \text{h.o.t. in } \beta$$

with

$$\frac{1}{f_0} = 1 + \left(\frac{\beta}{2}\right)^2\cos\theta + \frac{2}{3}\left(\frac{\beta}{2}\right)^4\cos^2\theta + \text{h.o.t. in } \beta$$

and substitute them together with the expression for η in (54) in (82). Then, it is a straightforward matter to show that the first leading-order term involving λ^a gives

$$\lambda^a = \frac{\alpha\gamma}{3}k^2(1-k^2),$$

which is in complete agreement with (85), with the corresponding eigenfunction G^a being

$$G^a = \frac{\alpha}{3}k^2(\cos\theta - 1).$$

Also, the next order in λ ($O(\beta^4)$) is zero, *i.e.*, $\lambda^b = 0$, with the corresponding eigenfunction G^b being

$$G^b = \frac{\alpha}{12}k^2(\cos 2\theta - 1).$$

Note that, according to Equation (85), the corresponding wavenumber k of the most rapidly growing disturbance equals $\sqrt{2}/2$ with the dimensional wavelength being $2R\sqrt{2}\pi$. Since the wavelength of such a disturbance is of the order of the cylinder radius, it appears likely that the mechanism for the instability, just presented, of a homogeneous liquid-film to axial disturbances due to surface tension, is closely related to that of the well-known Rayleigh instability of a stationary liquid-film coating the surface of a non-rotating cylinder in the absence of gravity. Specifically, the effects of surface tension, as expected, always tend to minimize the area of the air–liquid interface of the thin film, thereby *destabilizing* the thin film to axial disturbances but *stabilizing* it in the two-dimensional case. Furthermore, it is worth remarking at this point that Hosoi and Mahadevan [8] also examined this axial-stability problem by means of a numerical linear analysis starting from lubrication equations very similar to ours, but including inertia effects, and concluded that, for $F = 0{\cdot}06$ and $\gamma = 0{\cdot}5$, the flow should remain *stable* as long as inertial effects are weak. In view of our results, however, it would appear that some of the calculations of [8] could have been marred by numerical inaccuracies.

4.2. (b) Stability of the inhomogeneous film ($\beta > \beta^*$)

We begin with Equation (74) for the eigenvalue λ and the eigenfunction $G(\theta)$ and expand both $\eta_0(\theta)$ and $G(\theta)$ in a power series about $\theta = 0$, as was done earlier in the corresponding two-dimensional case. Then, we substitute these in (74). The first and second leading-order terms are exactly the same as those in the two-dimensional case, *i.e.*, Equations (67) and (68), and therefore the expression for the maximum eigenvalue λ is, as before,

$$\lambda = 2a_0a_1\frac{1 - \frac{5}{2}\alpha a_0}{1 - \alpha a_0} \cong -\sqrt{\frac{2}{3}} + O(\alpha) < 0.$$

Interestingly, this maximum eigenvalue λ is independent of k, the wavelength of the small axial disturbance, and is also essentially independent of β. This result shows that the two-dimensional steady solutions are asymptotically stable to axial perturbations for $\beta > \beta^*$, even for zero surface tension.

5. Discussion

First of all we have shown, on the basis of the modified lubrication analysis (MLA) leading to Equation (11) (which, we wish to stress once again, is a *model* equation, albeit a surprisingly accurate one [6]), that the discontinuous solutions of the SLE for $\beta > \beta^* \equiv 1{\cdot}414\ldots$ correctly represent the asymmetric film-thickness profiles but only under the restrictive condition: $\alpha \to 0$ with *β fixed*. On the other hand, as shown previously [6], the solutions to the standard lubrication equation fail to represent the film-thickness profiles when α is decreased with $F = \alpha\beta$ fixed, especially given that Equation (5) subject to (7) has no physically acceptable solutions when $\beta > 2{\cdot}21\ldots$.

Secondly, according to the stability analysis given by the modified lubrication equation for these two-dimensional rimming flows, we have found that, as shown previously [3, 10] on the basis of the standard lubrication theory, the *inhomogeneous* profiles are asymptotically stable to small two-dimensional as well as axial disturbances. In addition, the maximum eigenvalue of these asymmetric profiles (*i.e.*, the negative eigenvalue with the smallest absolute value) can be obtained simply by expanding the corresponding eigenequations about $\theta = 0$, rather than from a complete solution of these equations. It is surprising that this maximum eigenvalue is essentially independent of β and becomes equal to the maximum eigenvalue given by the standard lubrication equation as $\alpha \to 0$, even though the corresponding eigenfunctions are substantially different.

On the other hand, again in conformity with [3], the *homogeneous* profiles are found to be neutrally stable to small two-dimensional, as well as axial disturbances if surface-tension effects are neglected. When the latter are included in the stability analysis, however, the film thickness profiles are found to become asymptotically *stable* to small two-dimensional disturbances with the absolute value of the real part of the *least* stable eigenvalue increasing monotonically with β for $0 < \beta < \beta^*$, and to become asymptotically *unstable* to small, long-wavelength (i.e. $0 < k \leq O(1)$), *axial* disturbances with the growth rate of the most rapidly amplified disturbance increasing monotonically with β for $\beta < \beta^*$. This implies that, when surface tension is included in the analysis, the homogeneous profiles at the larger value of β should be more *stable* to two-dimensional disturbances, but should become more *unstable* to axial disturbances.

The theoretical predictions regarding the stability of *inhomogeneous* films are consistent with all the experimental results reported in the literature [3, 6, 16, 17]. For *homogeneous* films, however, the picture is less clear because, especially within the range $0{\cdot}7 < \beta < 1{\cdot}414$, the film profiles are often highly irregular and time-dependent due to a slight misalignment of the cylinder axis from the horizontal [3, 17, 18]. Clearly, additional experiments are required in order to test the characteristics of the homogeneous films.

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